Zeroes and $\mathbb{Q}$-points of smooth functions

Meeting ANR FOLIAGE

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- For $x = (x_1, \dots, x_n) \in \mathbb{Q}^n$, $x_i = \dfrac{a_i}{b_i}$ with $a_i \wedge b_i = 1$,

$$\boxed{\mathrm{ht}(x) := \max_i \{|a_i|, |b_i|\},}$$

- $X \subset \mathbb{R}^n$,

$$\boxed{X(\mathbb{Q}, T) := \{x \in X \cap \mathbb{Q}; \mathrm{ht}(x) \leq T\}}$$

- "X has few rational points of height $\leq T$" means subpolynomial bound:

$$\boxed{\forall \varepsilon > 0, \exists C_\varepsilon,\ \#X(\mathbb{Q}, T) \leq C_\varepsilon T^\varepsilon}$$

- "Better bound" means poly-log bound:

$$\boxed{\#X(\mathbb{Q}, T) \leq \alpha \log^\beta T}$$

- Bézout bound:

$$\boxed{Z_d(X) := \sup_{P \in \mathbb{R}_d[x] \setminus \{0\}} \#(X \cap P^{-1}(0)).}$$

Following Bombieri & Pila's approach, showing that a given set X has few rational points amounts:

- Yomdin-Gromov decomposition: For each $k \in \mathbb{N}$ covering X by a finite number of C^k-charts with all derivatives bounded by 1,
- Bézout bound: Showing that $Z_d(X)$ is uniformly bounded with respect to all nonzero polynomials P_d of degree $\leq d$.

Following Binyamini and Novikov's approach: control the number of Weierstraß polydiscs covering X through their size, using Bernstein indices in holomorphic-Noetherian setting or metric entropy measures in the Pfaffian context of $\mathbf{R}^{\mathrm{RE}}$.

Remark. For better bounds one needs to accurately control the number of charts and Z_d (polynomial in d).

For this, following Bombieri & Pila's strategy we can split difficulties in two:

1. Take X in a geometric structure where polynomial Bézout bounds are available, and work on the number of C^k-charts.
   (Binyamini & Novikov's approach for $\mathbf{R}^{\mathrm{RE}}$, C.-Miller for noncompact oscillatory "slow" sets).
2. Take X given by one convenient chart (e.g. a graph) and work on Z_d (C.-Yomdin, Villemot).

$$\boxed{\text{From now on, } X \text{ is a graph of a function or a parameterized curve.}}$$

- $f : D \to \mathbb{C}$ an analytic function (D: domain $/\mathbb{R}$ or $/\mathbb{C}$...),
- Γ_f the graph of f,
- $Z_d(f) := \sup_{P \in \mathcal{P}_d \setminus \{0\}} \#\{P(z, f(z)) = 0\} \in \mathbb{N} \cup \{\infty\}$.
- $Z_d(f)$ is the maximum number of intersection points between Γ_f and algebraic curves of degree $\leq d$.
- In case f is a polynomial, $Z_d(f)$ is polynomially bounded in d (and $\deg f$) when the intersection is transverse.
- $Z_d(f) < \infty$ or Γ_f contains a piece of algebraic curve (of dimension 1).

We assume $\forall d \in \mathbb{N}$, $Z_d(f) < \infty$, ie f is a *transcendental function*.

Sub-polynomial bounds

- [Bombieri & Pila 1989]: $f : [0,1] \to \mathbb{R}$ analytic then

$$\forall \varepsilon > 0 \; \exists C_{f,\varepsilon} \;\text{ s.t. }\; \#\Gamma_f(\mathbb{Q}, T) \le C_{f,\varepsilon} T^{\varepsilon}.$$

- [Pila & Wilkie 2006]: Same result for $X \subset \mathbb{R}^n$ an o-minimal set:

$$\forall \varepsilon > 0 \; \exists C_{X,\varepsilon} \;\text{ s.t. }\; \#X^{\text{trans}}(\mathbb{Q}, T) \le C_{X,\varepsilon} T^{\varepsilon}.$$

$X^{\text{trans}} := X \setminus X^{\text{alg}}$,
$X^{\text{alg}} := \{x \in X; \exists S \text{ semialgebraic of pure dimension } 1, \text{ s.t. } x \in S \subset X\}$.

Fix $T, d \geq 0, I \subset [0,1]$ and denote $\mu = \dfrac{(d+1)(d+2)}{2} = \dim(\mathbb{R}_{\leq d}[X,Y])$.

- If $\#\Gamma_f(\mathbb{Q},T) < \mu$, then $\Gamma_f(\mathbb{Q},T)$ is contained in a single set $P=0$, with $P \in \mathbb{R}_{\leq d}[X,Y]$. So we may assume $\#\Gamma_f(\mathbb{Q},T) \geq \mu$.
- Let $A_1,\dots,A_\mu \in \Gamma_f(\mathbb{Q},T)$, $A_i = (a_i, f(a_i))$, $A_i^{(\alpha,\beta)} = a_i^\alpha f^\beta(a_i)$,

$$M = (A_i^{(\alpha,\beta)})_{i\leq\mu,\alpha+\beta\leq d},$$

$$\Delta := \det(M).$$

- $$|\Delta| \leq C_d |I|^{\frac{\mu(\mu-1)}{2}} \qquad (*)$$
- On the other hand, if $\Delta \neq 0$, then $\prod_{i\leq\mu} |p_i^d q_i^d \Delta| \geq 1$, where p_i, q_i are the denominators of a_i and $f(a_i)$.

$$\Longrightarrow |\Delta| \geq T^{-2d\mu} \qquad (**)$$

- **Conclusion** $(*) + (**)$: $|I| \leq C_d' T^{\frac{-4d}{\mu-1}} \Longrightarrow \Delta = 0$.

Bombieri-Pila: Proof

- Assume for simplicity $f : [0,1] \to \mathbb{R}$.
- Let $I \subset [0,1]$, $|I| \leq C'_d T^{\frac{-4d}{\mu-1}}$. On needs $N \sim \dfrac{1}{C'_d} T^{\frac{4d}{\mu-1}}$ intervals I to cover $[0,1]$. Note that $\nu_d := \frac{4d}{\mu-1} \sim \frac{8}{d} \to 0$ as $d \to \infty$.
- For any choice of $A_1, \dots, A_\mu \in \Gamma_{f\restriction I}(\mathbb{Q}, T)$, $\Delta_{\restriction I} = 0$ thus

$$\operatorname{rank}(\mathrm{M}) < \mu.$$

  Let $A_1, \dots, A_\mu \in \Gamma_{f\restriction I}(\mathbb{Q}, T)$ realize the maximum rank r for M and let $\widetilde{M}$ a maximal minor in M (the upper left hand corner minor...). Let (α, β), $\alpha + \beta \leq d$, an exponent not appearing in $\widetilde{M}$.
- The non-zero polynomial $P_I(X,Y) \in \mathbb{R}_{\leq d}[X,Y]$

$$P_I(X,Y) := \det \begin{pmatrix} & & & X^{\alpha_1}Y^{\beta_1} \\ & & & X^{\alpha_2}Y^{\beta_2} \\ & \widetilde{M} & & \vdots \\ & & & X^{\alpha_r}Y^{\beta_r} \\ a_1^{\alpha} f(a_1)^{\beta} & \cdots & a_r^{\alpha} f(a_r)^{\beta} & X^{\alpha}Y^{\beta} \end{pmatrix}$$

  cancels at any point of $\Gamma_{f\restriction I}(\mathbb{Q}, T)$, as a minor of size $r+1$ of M.

Remark. [Pila 2006, Prop. 2.4]: $\Gamma_f(\mathbb{Q}, T)$ is contained in a certain number of hypersurfaces of $\mathbb{R}^2$ of degree d, this number being bounded by

$$C_{f,d} T^{\nu_d},$$

- $C_{f,d}$ depends on d and comes from analytic bounds on $|f^{(p)}|$ (linearly in A, $|f^{(k)}(x)/k!| \leq A^k$).
- for $d = \lfloor \log T \rfloor$, T^{ν_d} and C_d are constant independent of T.

Consequence 1 (Bombieri-Pila's end of proof).
For $\varepsilon > 0$ given, find $d := d_\varepsilon$ large enough s.t. $\nu_{d_\varepsilon} \leq \varepsilon$:

$$\#\Gamma_f(\mathbb{Q}, T) \leq Q(A, d_\varepsilon) \times Z_{d_\varepsilon}(f) \times T^\varepsilon. \qquad \square$$

Consequence 2 (possible better bound). For $f : [0, 1] \to \mathbb{R}$ analytic, transcendental

$$\#\Gamma_f(\mathbb{Q}, T) \leq C(A) \times Z_{\lfloor \log T \rfloor}(f).$$

In particular when f has a polynomial Bézout bound one gets a better bound

$$\exists \alpha, \beta \;\text{ s.t. }\; \#\Gamma_f(\mathbb{Q}, T) \leq \alpha \log^\beta T \quad (\text{vs} \quad \forall \varepsilon > 0,\; \#\Gamma_f(\mathbb{Q}, T) \leq \alpha C_\varepsilon T^\varepsilon)$$

Natural directions for better bounds

1. Non-compact domain. Find a class of $\mathcal{C}^\infty$ $f : [a, +\infty[\to \mathbb{R}$ with
   - $\forall p \geq 0$, fast enough decay for $|f^{(p)}|$.
   - $\forall d \geq 1, \forall I \subset [a, +\infty[$, polynomially controlled $Z_d(f_{|I})$ w.r.t. d and length(I),
2. Compact domain. Find analytic functions $f : D \to \mathbb{R}$ with $Z_d(f)$ polynomially bounded in d.

Better bound for $\#\Gamma_f(\mathbb{Q}, T)$

- (oscillatory) functions $f : [a, +\infty[\to \mathbb{R}$,
- more generally for possibly non compact and/or oscillatory curves $\Gamma \subset \mathbb{R}^n$.

Definition. The $\mathcal{C}^\infty$-parametrization $\gamma = (f, g) : [a, +\infty[\to \mathbb{R}^2$ of the curve $\Gamma \subset \mathbb{R}^2$ is *slow* when

1. $\exists u \in \mathbb{R}, \forall x,\ |u - f(x)| \leq b(x) \searrow 0,$
2. $\forall p \geq 0, \forall x,\ |\dfrac{f^{(p)}(x)}{p!}| \leq \varphi_p(x),\ |\dfrac{g^{(p)}(x)}{p!}| \leq \varphi_p(x)$, where

   $\exists$ constants A, B, C, D s.t. $\forall x,\ \varphi_p(x) = D\Big(Ap^B \dfrac{\log^C x}{x}\Big)^p.$

Remark. Functions satisfying 2 yield a subalgebra of $\mathcal{C}^\infty([a, +\infty[)$ stable under derivation.

Example. $g := h \circ \log^\ell$, where $\ell \geq 1$ and $\exists \alpha, \forall p \geq 0,\ |h^{(p)}(x)| \leq \alpha^p$.

Definition. $\varphi : [a, +\infty[\to \mathbb{R}$ is a *height control function* of γ when

$$\gamma^{-1}(\Gamma(\mathbb{Q}, T)) \subset [a, \varphi(T)].$$

Examples. For f slow:

- when $u \in \mathbb{Q}$ and f doesn't take the value u, one can take $\varphi(T) = b^{-1}(\dfrac{K}{T})$, for some K.
- When $u \notin \mathbb{Q}$, and is not a U-number of degree 1 in Mahler's classification, one can take $\varphi(T) = b^{-1}(\dfrac{1}{T^K})$, for some K.

Theorem. [C.-Miller 2017] Let γ be a slow parametrization of a transcendental curve Γ, with height control function φ, $T \geq 1$, $d \geq 1$ and

$$Z_{d,A} := \sup_{P \in \mathbb{R}_d[X,Y]\setminus\{0\}} \# P^{-1}(0) \cap \gamma([a,A]),$$

then

$$\#\Gamma(\mathbb{Q},T) \leq \alpha \log^{\delta}(T) \times \log^{\nu}(\varphi(T)) \times Z_{\log T, \varphi(T)}.$$

Consequence. When $e^{\varphi(T)}$ and $Z_{d,A}$ are polynomially bounded in T, d, A then

$$\#\Gamma(\mathbb{Q},T) \leq \alpha \log^{\beta}(T).$$

Examples. Built on elementary functions composed with $\log^{\ell}$
$\implies$ Slowness & $Z_{d,A}$ suitable ([Khovanskiĭ]).

- log-spirals: $(\frac{1}{x^F}\sin\circ\log^{\ell}, \frac{1}{x^G}\cos\circ\log^{q})$, $F, G > 0$, $\ell, q \in \mathbb{N}^*$.
- $(\log 2 + \frac{\arctan \log^2 x}{x^5(2+\cos^3 \log x)}, \pi + \frac{\sin \log^2 x}{\sqrt{x}(1+\log\log x)})$, here $b(x) = \frac{1}{x^5}$, $\varphi(T) = T^{\frac{1}{5}}$
- Graphs: $x \mapsto \sin \log^{\ell} x$ max. sol. of Euler equation $x^2 y'' + xy' + y = 0$.

$D = D(0,1)$ or $\mathbb{C}$ or $\bar{D}(0,1)$ etc.
$f : D \to \mathbb{C}$ transcendental analytic function.
How to prescribe polynomial bounds for $Z_d(f)$?

- $Z_d(f) \leq K_d < \infty$ holds for f in any o-minimal structure (of course for f transcendental...).
- On the other hand $Z_d(f)$ may be polynomially bounded in d while f is not o-minimal (see [Gwoździewicz-Kurdyka-Parusiński 1999]).
- Even when f is analytic, the asymptotic of $Z_d(f)$ is difficult to predict: for any $\zeta \in]0,1[$, there exists $f : D \to \mathbb{C}$ analytic such that for a sequence of degrees d going to ∞,

$$Z_d(f) \geq e^{d^\zeta}.$$

(see [Surroca 2002, 2006],[Pila 2004])

- For f entire of finite order $:= \limsup_{r\to\infty} \dfrac{\log\log \max_{D_r} |f|}{\log r} < \infty$, for a certain sequence of degrees going to ∞

$$Z_d(f) \leq Cd^2 \qquad \text{(best possible asymptotic)}.$$

What's known on the asymptotic of $Z_d(f)$?

- $f(z) = e^z$ has a polynomial Bézout bound:

  [Tijdeman 1971],

- Elementary functions have polynomial Bézout bounds:

  [Khovanskiĭ 1991],

- Entire functions with $0 <$ lower order $\leq$ finite order $< \infty$, have polynomial Bézout bounds:

  [Coman & Poletsky 2003, 2007],[Brudnyi 2008],[Boxall & Jones 2015].

- Specific functions like the Riemann ζ function, the Euler Γ function, have polynomial Bézout bounds:

  [Coman & Poletsky 2007],[Masser 2011],[Besson 2011, 2014],[Boxall & Jones 2013].

- (Compact) solutions of some algebraic differential equations have polynomial Bézout bounds:

  [Binyamini 2016].

Notation.

- $\Psi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ analytic,
- $Q_1, \cdots, Q_m : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ analytic maps.

  For $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{C}^n$, let
- $Q_\lambda(z) = \sum_{i=1}^m \lambda_i Q_i(\Psi(z)) = \sum_{k=0}^\infty v_k(\lambda) z^k$, $v_k(\lambda)$ linear forms on $\mathbb{C}^m$.
- $L_i := \{v_0 = v_1 = \cdots = v_i = 0\}$, $\mathbb{C}^m \supseteq L_0 \supseteq L_1 \supseteq \cdots \supseteq L_i \supseteq \cdots$

This sequence stabilizes at *the Bautin index $b = b_{\Psi, Q_1, \cdots, Q_m}$*:

$$L_{b-1} \supsetneq L_b = L_{b+1} = \cdots$$

Remark. $\lambda \in L_b \iff \forall k \geq 0,\ v_k(\lambda) = 0.$

Application.

- $n = 2$, $\Psi(z) = (z, f(z))$ $\; Q_i = X^j Y^p$, $\; j, p \in [\![0, d]\!]$, $\; m = (d+1)^2$,

$$\text{then } Q_\lambda(z) = \sum_{j=0}^{d} p_j(z) f^j(z), \ \deg p_j \leq d.$$

Remarks.

- The maximum number (w.r.t. λ) of zeroes of Q_λ bounds $Z_d(f)$.
- Since $\lambda \in L_b \iff \forall k \geq 0,\ v_k(\lambda) = 0$, when f is transcendental $\lambda \neq 0$ cannot cancel all v_k, therefore $L_b = \{0\}$.
- $\mathbb{C}^m \supsetneq L_0 \supsetneq \cdots \supsetneq L_b = \{0\}$,
  therefore b minimal for $m - 1 = d^2 + 2d$.
- But b may be very large!

Remarks.

- For $d \geq 1$
$$b = \max_{P \in \mathbb{C}_d \setminus \{0\}} \#\{\mathrm{mult}_0 P(z, f(z))\}$$

- $(b = b_d)_{d \geq 1}$ measures the transcendency of f: the faster $(b = b_d)_{d \geq 1}$ goes to ∞ the less f seems transcendental.

- $\lim_{r \to 0} \max_{P \in \mathbb{C}_d \setminus \{0\}} \#\{z \in D_r;\ P(z, f(z)) = 0\} \leq b.$

For a basis $(v_{i_1}, \cdots, v_{i_m}) \subset \{v_0, \cdots, v_b\}$ of the space of linear forms $\ell = \sum_{i=1}^m \alpha_i \lambda_i$ on $\mathbb{C}^m$ s.t. $\ell_{|L_b} \equiv 0$, write:

$$\ell = \sum_{q=1}^m \mu_q v_{i_q} \text{ and } \max_q |\mu_q| \leq c' \|\ell\|, \quad \text{with} \quad \|\ell(\lambda)\| = \max_i |\alpha_i|.$$

Notation. Let c be the minimum of the constant c' w.r.t. all possible choices of such basis.

Theorem. [Roytwarf & Yomdin 1997]

On $D_{\frac{1}{4}}$: $\boxed{Z_d(f) \leq 5b \log(4 + 2c(b+1)).}$

The bound $\boxed{Z_d(f) \leq 5b\log(4+2c(b+1))}$ comes from zero lemma for Berstein classes:

$h(z) = \sum_{i\geq 0} v_i z^i$ on $\bar{D}(0,R)$, for c and N s.t.

$$\forall j |v_j| R^j \leq c \max_{i\leq N} |v_i| R^i,$$

$$\boxed{\# h^{-1}(0) \cap D(0,R/4) \leq 5N + \log_{5/4}(2+c).}$$

Remark. In case $v_0 \neq 0$, one can always take $N=0$, and we get the classical Jensen estimate:

$$\# h^{-1}(0) \cap D(0,R/4) \leq \log_{5/4}(2 + \frac{M(h,R)}{|v_0|}).$$

Notation.

- $$a_i^j := \frac{1}{i!}(f^j)^{(i)}(0)$$
- After reduction, the matrix with lines the v_k's is:

$$M = \begin{pmatrix} a_{d+1}^1 & \text{---} & a_1^1 & \cdots & a_{d+1}^d & \text{---} & a_1^d \\ | & & | & & | & & | \\ a_b^1 & \text{---} & a_{b-d}^1 & \cdots & a_b^d & \text{---} & a_{b-d}^d \end{pmatrix} \quad (1)$$

- Δ the absolute value of a nonzero $(d^2+d) \times (d^2+d)$ minor of M. (exists since $L_b = \{0\}$!)

Theorem. [C.-Yomdin 2016] *On* $D_{\frac{1}{4}}$:

$$Z_d(f) \leq 5b \log(4 + 2(b+1)\frac{e^{2(d+1)^2 \log(d+1)}}{\Delta}).$$

Consequence. *When there exist* $R, S \in \mathbb{R}_+[X]$ *s.t.*

$$\forall d \in \mathbb{N}, \quad b \leq R(d) \quad \text{and} \quad \Delta \geq e^{-S(d)},$$

$Z_d(f)$ *is polynomially bounded on* $D_{\frac{1}{4}}$.

Notation. When $\forall k \geq 0,\ a_k = \dfrac{p_k}{q_k} \in \mathbb{Q}$ denote $h_\ell := \max\{|q_0|, \cdots, |q_\ell|\}$.

Proposition 1. [C.-Yomdin 2016] *For $f \in \mathbb{Q}\{z\}$, if there exist $R, S \in \mathbb{R}[X]$ s.t.*

$$b_d \leq R(d) \text{ and } h_\ell \leq e^{S(\ell)}$$

then $Z_d(f)$ is polynomially bounded.

Definition. f is *hypertranscendental* when f satisfies no algebraic differential equation.

Notation. For f hypertranscendental,

$$\eta_d := \max_{P \in \mathbb{Z}_d[X_0, \cdots, X_d] \setminus \{0\}} \{\mathrm{mult}_0 P(z, f(z), f'(z), \cdots, f^{(d)}(z))\}$$

Proposition 2. *For $f \in \mathbb{Q}\{z\}$, if there exist $R, S \in \mathbb{R}[X]$ s.t.*

$$\eta_d \leq R(d) \text{ and } h_\ell \leq e^{S(\ell)}$$

then f has a polynomial Bézout bound.

Theorem 1. [C.-Yomdin 2016] *Assume* $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \in \mathbb{C}\{z\} \setminus \mathbb{C}[z]$, *with* $n_k^2 \le n_{k+1} \le n_k^q$, *for some* $q > 2$, *then*

- f is hypertranscendental ([Ostrowski 1920]).
- $b_d \le d^{q^2}$.

  If furthermore $|a_k| \ge e^{-n_k^p}$, *for some* $p > 0$, *then*
- $Z_d(f) \le 10(2d)^{q^2}(1 + qd^2 + 5d^{pq+3})$.

Theorem 2. (cf also [Binyamini 2016]) *Assume* $f(z)$ *is a solution of an algebraic differential equation*

$$f^{(d)} = Q(z, f(z), \ldots, f^{(d-1)}(z)),\ Q \in \mathbb{Q}[X_1, \ldots, X_d],$$

with initial conditions in $\mathbb{Q}$. *Then* f *has a polynomial Bézout bound.*

Proof. b_d is polynomially bounded ([Nesterenko 88] or [Gabrielov 99]). The Taylor coefficients of f are rational and $h_\ell \le e^{S(\ell)}$.

Notation.

- $[0,1] = I \leftarrow \cdots \leftarrow I^n \leftarrow I^{n+1} \leftarrow \cdots \leftarrow I^\infty = \varprojlim_{n\in\mathbb{N}} I^n$
- $f = \sum_{k=0}^{\infty} a_k z^k \sim (a_k)_{k\in\mathbb{N}} \in I^\infty$
- μ the measure on I^∞ induced by cylinders $\pi_n^{-1}(G), G \in I^n$ μ_n-measurable, where $\mu(\pi_n^{-1}(G)) := \mu_n(G)$.

Theorem 3. *For μ-a.e. $f \in I^\infty$, $\exists U \in \mathbb{R}_{\leq 8}[X]$, $Z_d(f) \leq U(d)$.*

$\mathbb{Q}$-Theorem. *For* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, *assume one of the following conditions*

1. $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ s.t. $b_d \leq R(d)$, $h_l \leq e^{S(l)}$,
2. $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ s.t. $\eta_d \leq R(d)$, $h_l \leq e^{S(l)}$,
3. $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \in \mathbb{R}\{z\}$, $n_k^2 < n_{k+1} \leq n_k^q$, for $q > 2$, $|a_k| \geq e^{-n_k^p}$, for $p > 0$,
4. f is a solution of an algebraic differential equation with rational coefficients and initial conditions,
5. f is a random series,

then there exists $\alpha, \beta > 0$, s.t. $\#\Gamma_f(\mathbb{Q}, T) \leq \alpha \log^\beta T$.

Remarks.

- Statement 4 has been obtained in [Binyamini 2016].
- Statement 5 is a consequence of [Boxall & Jones 2013]: a_0 transcendent with 'good' transcendency measure, such as S-numbers in Mahler's classification (a.e. numbers are S-numbers) is enough for $\#\Gamma_f(\mathbb{Q}, T) \leq \alpha \log^\beta T$.

$\mathbb{Q}$-Theorem. *For* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, *assume one of the following conditions*

1. $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ *s.t.* $b_d \leq R(d)$, $h_l \leq e^{S(l)}$,
2. $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ *s.t.* $\eta_d \leq R(d)$, $h_l \leq e^{S(l)}$,
3. $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \in \mathbb{R}\{z\}$, $n_k^2 < n_{k+1} \leq n_k^q$, *for* $q > 2$, $|a_k| \geq e^{-n_k^p}$, *for* $p > 0$,
4. f *is a solution of an algebraic differential equation with rational coefficients and initial conditions,*
5. f *is a random series,*

then there exists $\alpha, \beta > 0$, *s.t.* $\#\Gamma_f(\mathbb{Q}, T) \leq \alpha \log^{\beta} T$. *($\beta = 8$ possible in 5.)*

Remark (Siegel-Shidlovskii's theorem). A combination of comparable - but finer - conditions on f than $1, 2, 4$, generalizing Lindemann-Weierstrass theorem (f E-function), gives

$$z \neq 0, z \in \bar{\mathbb{Q}} \Longrightarrow f(z) \text{ transcendental.}$$

In particular $\#\Gamma_f(\mathbb{Q}, T) \leq 1$.

PhD generalizing Coman & Poletsky to the meromorphic case and holomorphic case on $\mathbb{D}$.

$f \in \mathcal{M}(\mathbb{D})$, transcendental, Nevanlinna characteristic:

- $T(\nu, f) := \int_0^{2\pi} \log_+ |f(\nu e^{it})| \, \frac{\mathrm{d}t}{2\pi} + \int_0^{\nu} n(\infty, r) \, \frac{\mathrm{d}r}{r}$.
- $\mathrm{diam}_N(E) = \inf\{\sum_{i=1}^N r_i; E \subset \cup_{i=1}^N D(*, r_i)\}$

Theorem. Assume there exist (λ_d), $(\nu_d) \nearrow 1$ with

- $\log \frac{\nu_d}{\lambda_d} \geq d^{-\alpha}$,
- $T(\nu_d, f) \leq d^{\beta}$
- $\exists A_d \subset S^1$, $\mathrm{length}(A_d) \geq e^{-d^l}$, s.t.

  $\forall a \in A_d,\ \exists B_a \subset f^{-1}(a) \cap D(0, \lambda_d),\ \exists C_a \subset f^{-1}(\infty) \cap D(0, \nu_d)$,
  with $\mathrm{diam}_{d+\bar{n}(\nu_d,\infty)-\#C_a}(B_a, D(0, \nu_d)) \geq e^{-d^{\gamma}}$ and $p_{\nu_d}(B_a, C_a) \geq e^{-d^{\delta}}$,
  then

$$\boxed{Z_{d,z}(f) \leq \max(d, \phi(r))^{1+2\alpha+\max(\alpha,\gamma,\delta,l)}},$$

where $\phi(r) = \min\{k; \lambda_k \geq r\}$

Theorem. Let $f \in \mathcal{O}(\mathbb{D})$, transcendental, s.t.

$$\lambda := \liminf_{r\to 1^-} \frac{\log\log^+ M(f,r)}{-\log(1-r)} > 1, \text{ and } \rho := \limsup_{r\to 1^-} \frac{\log\log^+ M(f,r)}{-\log(1-r)} < \infty,$$

then

$$\boxed{\forall d \in \mathbb{N}, r \in [0,1[,\ Z_{d,r} \leq (\max(\log^+ M(f,r), d))^{\alpha_{\lambda,\rho}}.}$$

Example.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,\ z \in \mathbb{D},$$

with

$$\liminf_{n\to\infty} \frac{\log \lambda_n}{\log \lambda_n - \log^+ \log^+ |a_n|} > 2 \quad \text{and} \quad \liminf_{n\to\infty} \frac{\log \lambda_n}{\log \lambda_n - \log^+ \log^+ |a_n|} < \infty$$

For instance, $\boxed{\lambda_{n+1} = \phi(\lambda_n)}$, with $\phi(n) > n$ and $\boxed{a_n = e^{\lambda_n^{\frac{q-1}{q}}},\ q > 2.}$

Example. $f \in \mathcal{M}(\mathbb{C})$ elliptic (non constant),

$$\boxed{\exists C > 0, \forall d \in \mathbb{N}, r > 0,\ Z_{d,r} \leq C \max(d,r)^2.}$$

Same bound for σ (see also Besson) and for ζ (see also Jones-Thomas for its $\mathbb{Q}$-points, using the pfaffian structure - no zero lemma in their case) of Weierstraß

Example. Let $f \in \mathcal{M}(\mathbb{D})$ a nonconstant fuchsian function, then

$$\boxed{\forall d \in \mathbb{N}, \forall r \in [0,1[,\ Z_{d,r} \leq C \max\left(d^4, \frac{1}{(1-r)^4}\right).}$$